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# ON AXISYMMETRIC MOTION OF AN INCOMPRESSIBLE ELASTIC MEDIUM UNDER IMPACT LOADING

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*The method of matched asymptotic expansions was employed to obtain approximate solutions to the one-dimensional boundary-value problems of nonlinear dynamic elasticity theory of impact loading on the surface of a cylindrical cavity of an incompressible medium that causes antiplane motion or torsion of the medium. The expansion of the solution in the near-front region is based on solutions of evolution equations different from the equations for quasi-simple waves.*

**Key words:** *nonlinear elasticity, shock wave, perturbation method, evolution equation.*

**Introduction.** Shock waves in nonlinear elastic media have been the subject of extensive research (see, for example, [1–4]). Generally, the solution of boundary-value problems is complicated by the mutual effect of volume and shear deformations and by uncertainty in the position and geometry of shock-wave surfaces, on which some boundary conditions are specified. Therefore, the solutions behind the frontal surfaces of shock waves and the shock position and geometry are interrelated and should be determined jointly. The nonlinear nature of the differential equations and boundary conditions makes it impossible to obtain exact solutions, and, hence, the role of numerical and approximate analytical methods increases. The best-known analytical methods are the radial method and the perturbation method [5, 6]. The latter was used in solutions of many problems. These are mainly problems that study deformations resulting in volume changes [7] or plane one-dimensional waves [6–8]. To analyze shear deformation separately from volume deformation, it is necessary to use a special model of a nonlinear elastic incompressible medium.

In the present paper, we consider two types of transverse waves propagating from a loaded cylindrical cavity: an antiplane deformation wave and a wave caused by a twisting impact. The solution is constructed using the method of matched asymptotic expansions. For each type of deformation, the solution of the inner problem is determined by the evolution equation. The obtained near-front expansions of the solutions can be used independently or included as initial data in numerical calculation schemes [9].

**Basic Model Relations of Nonlinear Elastic Incompressible Media.** In curvilinear Euler coordinates $x^1$, $x^2$, and $x^3$, the equations of the dynamics of an incompressible elastic medium are written as

$$v^i = \dot{u}^i + u^i_{,j} v^j, \qquad \alpha_{ij} = (u_{i,j} + u_{j,i} - u_{k,i} u^k_{,j})/2,$$

$$\sigma^{ij}_{,j} = \rho(\dot{v}^i + v^i_{,j} v^j), \qquad \sigma^i_j = -p\delta^i_j + \frac{\partial W}{\partial \alpha^k_i}\,(\delta^k_j - 2\alpha^k_j),$$

$$W = (a-\mu)I_1 + aI_2 + bI_1^2 - \chi I_1 I_2 - \theta I_1^3 + cI_1^4 + dI_2^2 + kI_1^2 I_2 + \ldots\ , \qquad (1)$$

$$I_1 = \alpha^j_j, \qquad I_2 = \alpha^i_j \alpha^j_i,$$

$$\dot{u}^i = \frac{\partial u^i}{\partial t}, \qquad u^i_{,j} = \frac{\partial u^i}{\partial x^j} + \Gamma^i_{kj} u^k, \qquad u_{i,j} = \frac{\partial u_i}{\partial x^j} - \Gamma^k_{ij} u_k,$$

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where $u^i$ and $v^i$ are the components of the displacement and velocity vectors of points of the medium, $\alpha_{ij}$ are the covariant components of the Almansi strain tensor, $\sigma^{ij}$ are the contravariant components of the Euler–Cauchy stress tensor, $\rho = \text{const}$ is the density of the medium, $W$ is an elastic potential function that specifies the properties of the medium, $p$ is the additional hydrostatic pressure, $a$, $b$, $\chi$, $\theta$, $c$, $d$, and $k$ are the elastic moduli of the medium; the subscript after the comma denotes covariant differentiation; the summation is performed over repeated Latin indices. System (1) corresponds to the motion of an isotropic medium in the adiabatic approximation. In the series expansion of $W$ in the neighborhood of the free state, some terms have the minus sign because for an incompressible medium, $I_1 < 0$ and $I_2 > 0$.

**Problem of Antiplane Impact Loading of an Incompressible Nonlinear Elastic Medium: Equations and Boundary Conditions.** We consider the solution of the problem of antiplane impact on a cylindrical circumferential cavity located in space occupied by an incompressible elastic medium. It is convenient to solve the problem in cylindrical coordinates $x^1 = r$, $x^2 = \varphi$, and $x^3 = z$. In this case, the equation of the cavity surface has the form $r = r_0$ and the medium occupies part of the space for which $r \geq r_0$. The radius $r_0$ is much smaller than the length of the generatrix; therefore, the generatrices are assumed to have an infinite length. Beginning at the time $t = 0$, the boundary of the cavity is acted upon by an impact load resulting in the occurrence of a displacement field $u_r = u_\varphi = 0$, $u_z = u_z(r, t)$, where for $r = r_0$,

$$u_z\Big|_{r=r_0} = U(t) = v_0 t + at^2/2, \qquad v_0 = \text{const} > 0, \qquad a = \text{const}. \tag{2}$$

Here, without loss of generality, we use a quadratic approximation for $U(t)$ to simplify the further presentation. By virtue of the condition $v_0 > 0$, a shock wave — a discontinuity surface of the displacement gradient — is separated from the loaded surface at the moment $t = 0$. It is known that on this surface, Eqs. (1) are not satisfied. On the discontinuity surface, the dynamic compatibility conditions implied by the integral conservation laws along with geometrical and kinematic compatibility relations should be satisfied [10, 11]. From these conditions, one can obtain the general form of the dependence of shock-wave velocity $G$ on its intensity and the initial strains in the medium [4, 5]. Assuming that initial strains are absent in the problem considered, we obtain the following approximation for the velocity $G$:

$$G \approx C(1 + \beta\tau^2 + \ldots), \qquad \tau\Big|_{r=r_0+\tilde{r}(t)} = \Big[\frac{\partial u_z}{\partial r}\Big] = -\frac{\partial u_z^-}{\partial r},$$
$$C = \sqrt{\frac{\mu}{\rho}}, \qquad \beta = \frac{a + b + \chi + d}{\mu}, \qquad \tilde{r}(t) = \int_0^t G(\xi)\, d\xi, \qquad [f] = f^+ - f^-. \tag{3}$$

Here square brackets indicate the jump of the quantity on the discontinuity surface, $f^+$ abd $f^-$ are the limiting values of $f$ (the superscript plus corresponds to the region to which the wave moves), and the quantity $\tau$ determines the wave intensity. We note that this wave is transverse. Below, the wave surface will be denoted by $\Sigma$. On the surface $\Sigma$, it is necessary to impose the following conditions:

$$u_z\Big|_\Sigma = 0, \qquad \tau\Big|_\Sigma = -\frac{\partial u_z^-}{\partial r}, \qquad [\sigma_{rr}]\Big|_\Sigma = 0, \qquad p^+ = p_0 = \text{const}. \tag{4}$$

The first condition in (4) is a consequence of the continuity of the displacement field, the second condition specifies the absence of initial strains, and the last two conditions allow the function $p$ to be determined. In the problem considered, Eqs. (1) imply the system

$$\frac{\partial^2 u_z}{\partial r^2}\Big(1 + 3\gamma\Big(\frac{\partial u_z}{\partial r}\Big)^2\Big) + \frac{1}{r}\,\frac{\partial u_z}{\partial r}\Big(1 + \gamma\Big(\frac{\partial u_z}{\partial r}\Big)^2\Big) + \ldots = \frac{1}{C^2}\,\frac{\partial^2 u_z}{\partial t^2},$$
$$\frac{1}{\mu}\,\frac{\partial p}{\partial r} = 2\alpha\,\frac{\partial u_z}{\partial r}\,\frac{\partial^2 u_z}{\partial r^2} - \frac{a + \mu}{r}\Big(\frac{\partial u_z}{\partial r}\Big)^2 + \ldots\,, \tag{5}$$
$$\gamma = (a + b + \chi + d)/\mu, \qquad \alpha = a - b - \mu - \chi/4.$$

From (5) it follows that the function $p$ depends on the displacement field, whose construction is the primary goal.

**Solution of the Problem of Antiplane Impact Using the Method of Matched Asymptotic Expansions.** To obtain approximate functional dependences for the field, we introduce the dimensionless variables

$$s = \frac{r - r_0}{r_0}\, \varepsilon^{-4}, \qquad m = \frac{r - r_0 - Ct}{r_0}\, \varepsilon^{-3}, \qquad w(s,m) = \frac{u_z}{r_0}\, \varepsilon^{-9/2}, \qquad \varepsilon = \Big(\frac{v_0}{C}\Big)^{2/3}. \tag{6}$$

Here $\varepsilon$ is a small parameter of the problem; the dependence of $s$ on $r - r_0$ implies that the solution is constructed in a region adjacent to the loaded cavity; the dependence of $m$ on $r - r_0 - Ct$ defines the near-front expansion of the solution.

Let us consider the displacement field in the near-front region at times close to $t = 0$. Substitution of (6) into (2) and (5) yields the following boundary-value problem for $w(s, m)$:

$$\begin{gathered}(w_{,ss} + 2\varepsilon w_{,sm})(1 + 3\gamma\varepsilon(w_{,s} + \varepsilon w_{,m})^2) + 3\gamma\varepsilon^3 w_{,mm}(w_{,s} + \varepsilon w_{,m})^2 \\ + \frac{\varepsilon^4}{1 + s\varepsilon^4}\,(w_{,s} + \varepsilon w_{,m}) + \frac{\gamma\varepsilon^5}{1 + s\varepsilon^4}\,(w_{,s} + \varepsilon w_{,m})^3 + \ldots = 0, \\ w|_{s=0} = -m + a_1\varepsilon^3 m^2/2, \qquad a_1 = a r_0/(v_0 C).\end{gathered} \tag{7}$$

The function $w(s, m)$ is represented as the asymptotic series

$$w(s,m) = w_0(s,m) + \varepsilon w_1(s,m) + \varepsilon^2 w_2(s,m) + \varepsilon^3 w_3(s,m) + \ldots\ . \tag{8}$$

Substituting expansion (8) into (7), with accuracy up to the third order of smallness in $\varepsilon$, we obtain

$$\begin{gathered}w(s,m) = f_0(m)s - m + \varepsilon(-f_0'(m)s^2 + f_1(m)s) \\ + \varepsilon^2(2f_0''(m)s^3/3 - f_1'(m)s^2 + f_2(m)) + \varepsilon^3(-f_0'''(m)s^4/3 - \gamma f_0^2(m)f_0''(m)s^3/2 \\ + 2f_1''(m)s^3/3 - f_2'(m)s^2 + f_3(m)s + a_1 m^2/2) + \ldots\ .\end{gathered} \tag{9}$$

Solution (9) is obtained ignoring the conditions on $\Sigma$. This solution includes the unknown functions $f_0$, $f_1$, $f_2$, and $f_3$, which are determined from an additional solution that satisfies the conditions at the leading edge of the wave. Series (9) will be called the outer expansion of the solution [12], and the additional solution will be called the inner expansion. Let us consider transition to the inner region in which the near-front type of solution is conserved in the neighborhood of $r = r_0$. This can be achieved by changing the scale $s$ and introducing the variable $p = \varepsilon^k s$ ($k = 1, \ldots, 4$). For $k = 1, 2, 3$, we obtain inner problems (by using the method of successive nonlinear approximations) and a linear wave operator in the zero step. To take into account the nonlinear nature of the problem, we consider the parameter $p = \varepsilon^4 s$. In the variables $p$, $m$, and $w(p, m)$, Eq. (5) is written as

$$\begin{gathered}(2w_{,pm} + \varepsilon^3 w_{,pp})\{1 + 3\gamma\varepsilon^3(\varepsilon^3 w_{,p} + w_{,m})^2\} + 3\gamma w_{,mm}(\varepsilon^3 w_{,p} + w_{,m})^2 \\ + \frac{1}{1+p}\,(\varepsilon^3 w_{,p} + w_{,m}) + \frac{1}{1+p}\,\gamma\varepsilon^3(\varepsilon^3 w_{,p} + w_{,m})^3 + \ldots = 0.\end{gathered} \tag{10}$$

From boundary conditions (4), we obtain

$$w(m,p)\Big|_{m=y(p)} = 0, \qquad \tau\Big|_{m=y(p)} = -\varepsilon^{3/2}(w_{,m} + \varepsilon^3 w_{,p}). \tag{11}$$

Here $y(p)$ is an unknown function that defines the position of the leading edge of the wave. From (11), it follows that at the distances considered, the shock wave has low intensity: $\tau \sim \varepsilon^{3/2}$. We represent the required inner solution as the series

$$w(m,p) = w_0(m,p) + \varepsilon^3 w_1(m,p) + \varepsilon^6 w_2(m,p) + \ldots\ , \tag{12}$$

in which the form of the calibration functions is given in accordance with (10). The function $y(p)$ is replaced by its approximate representation:

$$y(m,p) = y_0(m,p) + \varepsilon^3 y_1(m,p) + \varepsilon^6 y_2(m,p) + \ldots\ .$$

Substitution of (12) into (10) in the $k$th step yields the equation

$$w_{k,pm} + \frac{3\gamma}{2}\, w_{k,pm} w_{0,m}^2 + \frac{w_{k,m}}{2(1+p)} = F_k(m, p, w_{0,m}, \ldots, w_{k-1,m}, \ldots, w_{k-1,pm}), \tag{13}$$

where the functions $F_k$ are defined via $F_{k-1}$ and $F_0 = 0$. From (13) it follows that the main operator of the zero step is the evolution equation of cylindrical transverse waves:

$$w_{0,pm} + \frac{3\gamma}{2} w_{0,mm} w_{0,m}^2 + \frac{w_{0,m}}{2(1+p)} = 0. \tag{14}$$

We note the following difference between the evolution equation (14) for the deformations leading to shape changes and the equation of quasi-simple waves, which is the evolution equation for propagation of volume deformations: Equation (14) includes the quantity $w_{0,m}$ raised to the second rather than first power. The last term in (14) indicates that the wave is cylindrical.

For the position $\Sigma$, we obtain the ordinary differential equation

$$y'(p) = (\gamma/2)(\varepsilon^3 w_{,p} + w_{,m})^2 - (3/8)\gamma^2\varepsilon^3(\varepsilon^3 w_{,p} + w_{,m})^4 + \dots\ , \tag{15}$$

where $w_{,m}$ and $w_{,p}$ are functions of $p$ and $m$, respectively $[m = y(p)]$. The boundary condition for Eq. (15) is written as $y(0) = 0$. Simultaneous solution of Eqs. (13) and (15) subject to boundary conditions (11) yields the inner representation of the solution up to the third order in $\varepsilon$, which contains unknown constants.

Performing the standard procedure of matching the inner and outer expansions, which allows the unknown functions and constants to be determined, we obtain a uniformly suitable expansion of the solution; in the dimensionless variables, it has the form

$$\begin{aligned}
w = {} & -\frac{m}{(1+p)^{1/2}} + \frac{\gamma}{2}\frac{\ln(1+p)}{(1+p)^{1/2}} + \varepsilon^3\Big\{ -\frac{m^2}{16}\frac{1}{(1+p)^{3/2}} \\
& + \Big(\frac{5}{8}\gamma\frac{1}{(1+p)^{3/2}} + \frac{1}{16}\gamma\frac{\ln(1+p)}{(1+p)^{3/2}} - \frac{5}{8}\frac{\gamma}{(1+p)^{1/2}}\Big)m \\
& - \gamma\Big(\frac{1}{2} - \frac{5}{8}\gamma\Big)\exp\Big(\frac{3}{8}\gamma\frac{1}{1+p}\Big)\frac{\ln(1+p)}{(1+p)^{1/2}} - \frac{10}{3}\gamma\frac{1}{(1+p)^{1/2}} \\
& - \gamma\Big(\frac{1}{2} - \frac{5}{8}\gamma\Big)\exp\Big(\frac{3}{8}\gamma\frac{1}{1+p}\Big)\frac{1}{(1+p)^{1/2}}\sum_{n=1}^{\infty}(-1)^{n+1}\Big(\frac{3}{8}\gamma\frac{1}{1+p}\Big)^n\frac{1}{nn!} \\
& + \frac{\gamma}{2}\Big(1 + \frac{5}{8}\gamma\Big)\frac{\ln(1+p)}{(1+p)^{1/2}} + \frac{10}{3}\gamma\exp\Big(-\frac{3}{8}\gamma\frac{p}{1+p}\Big)\frac{1}{(1+p)^{1/2}} \\
& + \gamma\Big(\frac{1}{2} - \frac{5}{8}\gamma\Big)\exp\Big(\frac{3}{8}\gamma\frac{1}{1+p}\Big)\frac{1}{(1+p)^{1/2}}\sum_{n=1}^{\infty}(-1)^{n+1}\Big(\frac{3}{8}\gamma\Big)^n\frac{1}{nn!} \\
& + \frac{7}{32}\gamma^2\frac{\ln^2(1+p)}{(1+p)^{3/2}} - \frac{5}{16}\gamma^2\frac{\ln(1+p)}{(1+p)^{3/2}}\Big\}.
\end{aligned} \tag{16}$$

The position of the leading edge $\Sigma$ is given by the equation

$$\begin{aligned}
y(p) = {} & \frac{\gamma}{2}\ln(1+p) + \varepsilon^3\Big\{ -\frac{10}{3}\gamma - \gamma\Big(\frac{1}{2} - \frac{5}{8}\gamma\Big)\exp\Big(\frac{3}{8}\gamma\frac{1}{1+p}\Big)\ln(1+p) \\
& - \gamma\Big(\frac{1}{2} - \frac{5}{8}\gamma\Big)\exp\Big(\frac{3}{8}\gamma\frac{1}{1+p}\Big)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{nn!}\Big(\frac{3}{8}\gamma\frac{1}{1+p}\Big)^n \\
& + \frac{\gamma}{2}\ln(1+p) + \frac{10}{3}\gamma\exp\Big(-\frac{3}{8}\gamma\frac{p}{1+p}\Big) \\
& + \gamma\Big(\frac{1}{2} - \frac{5}{8}\gamma\Big)\exp\Big(\frac{3}{8}\gamma\frac{1}{1+p}\Big)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{nn!}\Big(\frac{3}{8}\gamma\Big)^n\Big\}.
\end{aligned} \tag{17}$$

**Problem of Twisting Impact on a Cylindrical Cavity in a Nonlinear Elastic Incompressible Medium.** We consider the one-dimensional problem of twisting impact on the inner surface of a cavity. The impact sets all points of the medium in motion on circles. For a linear elastic medium, this motion is modeled by one component of the displacement vector $u_\varphi(r;t)$ ($u_r = u_z = 0$). In the nonlinear model for describing the motion of the medium, we introduce the following displacement components:

$$u_r = r(1-\cos\Psi), \qquad u_\varphi = r\sin\Psi, \qquad u_z = 0$$

[$\Psi(r,t)$ is the rotation angle of points of the medium]. Solution of the problem reduces to determining the functions $\Psi(r,t)$ and $p(r,t)$ and the position of the discontinuity surface resulting from the impact. The following boundary conditions are imposed. On the surface being loaded, the condition is written as

$$\Psi\Big|_{\substack{r=r_0\\ t\geqslant 0}} = \varphi_0 t + \frac{\dot\varphi_0 t^2}{2}, \tag{18}$$

where $\varphi_0$ and $\dot\varphi_0$ are the angular velocity and angular acceleration. Constant rotation acceleration is chosen to simplify the relations resulting from the solution. In the absence of initial deformations, the following conditions hold at the shock wave:

$$\Psi\Big|_{r=r_0+\tilde r(t)} = 0, \qquad \tau\Big|_{r=r_0+\tilde r(t)} = -\frac{\partial\Psi^-}{\partial r}, \qquad [\sigma_{rr}]\Big|_{r=r_0+\tilde r(t)} = 0,$$

$$\tilde r(t) = \int\limits_0^t G(\xi)\,d\xi, \qquad G = C\Big(1+\frac{\gamma}{2}\,(r\tau)^2+\ldots\Big), \qquad p^+ = p_0 = \text{const}.$$

For the problem considered, the equations of motion are written as

$$\begin{gathered}\frac{\partial^2\Psi}{\partial r^2}\Big(1+3\gamma\Big(\frac{\partial\Psi}{\partial r}\Big)^2\Big) + \frac{1}{r^2}\Big(3r\,\frac{\partial\Psi}{\partial r} + 5\gamma r^3\Big(\frac{\partial\Psi}{\partial r}\Big)^3\Big) = \frac{1}{C^2}\,\frac{\partial^2\Psi}{\partial t^2} + \ldots\ , \\ \frac{1}{\mu}\,\frac{\partial p}{\partial r} + (\beta+1)r\Big(\frac{\partial\Psi}{\partial r}\Big)^2 + \beta r^2\,\frac{\partial\Psi}{\partial r}\,\frac{\partial^2\Psi}{\partial r^2} = \frac{1}{C^2}\,r\Big(\frac{\partial\Psi}{\partial t}\Big)^2 + \ldots\ .\end{gathered} \tag{19}$$

As above, the primary goal in constructing the solution is to find the field of the function $\Psi(r,t)$.

**Perturbation Method in the Problem of a Twisting Impact.** Assuming that the impact results in displacements of the same order of smallness as the displacements in the plane problem considered above, we specify the dimensionless variables of the outer problems in the near-front region:

$$\begin{gathered} s = \frac{r-r_0}{r_0}\,\varepsilon^{-4}, \qquad m = \frac{r-r_0-Ct}{r_0}\,\varepsilon^{-3}, \\ \varkappa(s,m) = \Psi\varepsilon^{-9/2}, \qquad \varepsilon = (v_0/C)^{2/3}, \qquad v_0 = r_0\varphi_0. \end{gathered} \tag{20}$$

Substituting (20) into (19) and (18), for the function $\varkappa$ we obtain the outer boundary-value problem

$$\begin{gathered}(\varkappa_{,ss} + 2\varepsilon\varkappa_{,sm} + \varepsilon^2\varkappa_{,mm})\{1 + 3\gamma\varepsilon(1+\varepsilon^4 s)^2(\varkappa_{,s}+\varepsilon\varkappa_{,m})^2\} \\ + \frac{\varepsilon^4}{(1+s\varepsilon^4)^2}\,\{3(1+\varepsilon^4 s)(\varkappa_{,s}+\varepsilon\varkappa_{,m}) + 3\gamma\varepsilon(1+\varepsilon^4 s)^3(\varkappa_{,s}+\varepsilon\varkappa_{,m})^3\} + \ldots = \varepsilon^2\varkappa_{,mm}, \\ \varkappa\Big|_{s=0} = -m + \varepsilon^3 K m^2/2, \qquad K = \dot\varphi_0 r_0^2/(Cv_0).\end{gathered} \tag{21}$$

Representing $\varkappa$ as the asymptotic series

$$\varkappa(s,m) = \varkappa_0(s,m) + \varepsilon\varkappa_1(s,m) + \varepsilon^2\varkappa_2(s,m) + \varepsilon^3\varkappa_3(s,m) + \ldots$$

and substituting this series into (21), we obtain an outer expansion of the solution for $\varkappa$ that is similar to (9), at least, up to the third order in $\varepsilon$. This expansion also contains unknown functions. To construct a uniformly suitable expansion, we pass into the inner region assuming that $p = \varepsilon^4 s$. In the variables $p$, $m$, and $\varkappa$, we obtain the inner boundary-value problem

$$\varkappa_{k,pm} + \frac{3\gamma}{2}(1+p)^2\varkappa_{k,mm}\varkappa_{0,m}^2 + \frac{3}{2(1+p)}\varkappa_{k,m} = H_k(m,p) \qquad (k=3i, \quad i=0,1,2,\ldots),$$
$$r\tau|_{m=y(p)} = -(1+p)\varepsilon^{3/2}(\varkappa_{,m} + \varepsilon^3\varkappa_{,p}), \qquad \varkappa(m,p)|_{m=y(p)} = 0, \tag{22}$$

where $y(p)$ is an unknown function that specifies the position of the leading edge of the shock wave; the function $\varkappa(m,p)$ is represented as an asymptotic series of the form (12); the functions $H_k(m,p)$ are defined via $H_{k-1}(m,p)$, and $H_0 = 0$. From (22), it follows that in the zero step, the solution of the problem is defined by the evolution equation

$$\varkappa_{0,pm} + \frac{3\gamma}{2}(1+p)^2\varkappa_{0,mm}\varkappa_{0,m}^2 + \frac{3\varkappa_{0,m}}{2(1+p)} = 0. \tag{23}$$

We note that (23) is not an independent evolution equation. Indeed, making the change of the unknown function by setting $w_0 = (1+p)\varkappa_0$, we obtain Eq. (14) for $w_0$ again. This function corresponds to the approximate solution $u_\varphi \approx r\Psi$, $u_r \approx 0$. The position $\Sigma$ is determined by solving the ordinary differential equation

$$y'(p) = \gamma(1+p)^2(\varkappa_{,m} + \varepsilon^3\varkappa_{,p})^2/2 - \gamma^2\varepsilon^3(1+p)^4(\varkappa_{,m} + \varepsilon^3\varkappa_{,p})^4/4 + \ldots\ , \tag{24}$$

where it is taken into account that $\varkappa = \varkappa(p,m)$ and $m = y(p)$. The boundary condition for Eq. (24) has the form $y(0) = 0$. Simultaneous solution of Eqs. (22) and (24) yields a solution for $\varkappa(p,m)$ up to the third order of smallness. The uniformly suitable solution on its basis is given by

$$\begin{aligned}
\varkappa = {} & -\frac{m}{(1+p)^{3/2}} + \frac{\gamma}{2}\frac{\ln(1+p)}{(1+p)^{3/2}} + \varepsilon^3\Big\{\frac{3m^2}{16}\frac{1}{(1+p)^{5/2}} \\
& + \Big(\frac{25}{8}\gamma\frac{1}{(1+p)^{5/2}} - \frac{3}{16}\gamma\frac{\ln(1+p)}{(1+p)^{5/2}} - \frac{25}{8}\gamma\frac{1}{(1+p)^{3/2}}\Big)m \\
& - \gamma\Big(-\frac{25}{8}\gamma + \frac{1}{2}\Big)\exp\Big(\frac{15}{8}\gamma\frac{1}{1+p}\Big)\frac{\ln(1+p)}{(1+p)^{3/2}} \\
& - \frac{31}{15}\frac{1}{(1+p)^{3/2}} + \frac{25\gamma^2}{16}\frac{\ln(1+p)}{(1+p)^{5/2}} + \frac{3}{64}\gamma^2\frac{\ln^2(1+p)}{(1+p)^{5/2}} \\
& + \gamma\Big(-\frac{25}{8}\gamma + \frac{1}{2}\Big)\exp\Big(\frac{15}{8}\gamma\frac{1}{1+p}\Big)\frac{1}{(1+p)^{3/2}}\sum_{n=1}^{\infty}\Big(-\frac{15}{8}\gamma\frac{1}{1+p}\Big)^n\frac{1}{nn!} \\
& - \gamma\Big(-\frac{25}{8}\gamma + \frac{1}{2}\Big)\exp\Big(\frac{15}{8}\gamma\frac{1}{1+p}\Big)\frac{1}{(1+p)^{3/2}}\sum_{n=1}^{\infty}\Big(-\frac{15}{8}\gamma\Big)^n\frac{1}{nn!} \\
& + \frac{31}{15}\gamma\exp\Big(-\frac{15}{8}\gamma\frac{p}{1+p}\Big)\frac{1}{(1+p)^{3/2}}\Big\}.
\end{aligned} \tag{25}$$

For $y(p)$, we obtain

$$\begin{aligned}
y(p) = {} & \frac{\gamma}{2}\ln(1+p) + \varepsilon^3\Big\{-\gamma\Big(\frac{1}{2} - \frac{25}{8}\gamma\Big)\exp\Big(\frac{15}{8}\gamma\frac{1}{1+p}\Big)\ln(1+p) \\
& + \frac{\gamma}{2}\ln(1+p) + \gamma\Big(\frac{1}{2} - \frac{25}{8}\gamma\Big)\exp\Big(\frac{15}{8}\gamma\frac{1}{1+p}\Big)\sum_{n=1}^{\infty}\frac{1}{nn!}\Big(-\frac{15}{8}\gamma\frac{1}{1+p}\Big)^n \\
& + \frac{31}{15}\exp\Big(-\frac{15}{8}\gamma\Big)\exp\Big(\frac{15}{8}\gamma\frac{1}{1+p}\Big) - \frac{31}{15} \\
& - \gamma\Big(\frac{1}{2} - \frac{25}{8}\gamma\Big)\exp\Big(\frac{15}{8}\gamma\frac{1}{1+p}\Big)\sum_{n=1}^{\infty}\frac{1}{nn!}\Big(-\frac{15}{8}\gamma\Big)^n\Big\}.
\end{aligned} \tag{26}$$

The final approximate solutions can be written in dimensional variables according to Eqs. (16), (17), (25), and (26).

In conclusion, we note the following difference in the nature of propagation of the deformations resulting in volume and shape changes: Eqs. (14) and (23) contain the quantities $w_{0,m}$ and $\varkappa_{0,m}$ raised to the second power, whereas the equation for quasi-simple waves contain these quantities raised to the first power. Obviously, this difference is retained in the Burgers type evolution equation if the dissipative factors are taken into account, and it is retained in the Korteweg–de Vries type evolution equation if dispersion is taken into account.

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